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## *SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. V.*

SOLUTIONS of problems in No. 6, Vol. V, have been received as follows: From R. J. Adcock, 239; Marcus Baker, 234; Prof. P. E. Chase, 239; George Eastwood, 232; F. H. Hoffstrom, 234; W. E. Heal, 234; Prof. W. W. Johnson, 236, 240; Chas. H. Kummell, 237, 239; Prof. Joseph H. Kershner, 232, 233, 234, 235, 237, 239; Prof. D. J. Mc Adam, 234, 237; Artemas Martin, 238; Prof. Orson Pratt, Sen., 231; P. Richardson, 234; E. B. Seitz, 232, 234, 236, 237, 238, 240; Prof. J. Scheffer, 232, 233, 234, 237; Prof. M. C. Stevens, 234.

231. "What is the sum expressed in terms of $m$, of the values of all the determinants, from the second to the $n$th orders inclusive, which can be formed from the $m$-gonal series of numbers, represented by $1, a_2, a_3, a_4, \ldots a_n$, the arrangement of the constituents of the respective determinants being after the following form:

$$\begin{vmatrix} 1, & a_2 \\ a_3, & a_4 \end{vmatrix} + \begin{vmatrix} 1, & a_2, & a_3 \\ a_4, & a_5, & a_6 \\ a_7, & a_8, & a_9 \end{vmatrix} + \begin{vmatrix} 1, & a_2, & a_3, & a_4, \\ a_5, & a_6, & a_7, & a_8, \\ a_9, & a_{10}, & a_{11}, & a_{12}, \\ a_{13}, & a_{14}, & a_{15}, & a_{16}, \end{vmatrix} + \text{\&c.?"}$$

SOLUTION BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.

The value of the determinant of the second order, in terms of $m$, is

$$\begin{vmatrix} 1, & m \\ 3m-3, & 6m-8 \end{vmatrix} = -3m^2+9m-8.$$

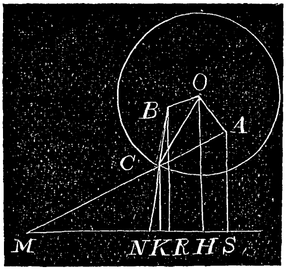
That of the third order is

$$\begin{vmatrix} 1, & m, & 3m-3 \\ 6m-8, & 10m-15, & 15m-24 \\ 21m-35, & 28m-48, & 36m-63 \end{vmatrix} = -3^3(m-2)^3.$$

All determinants of higher orders vanish; therefore the sum of these two values is the solution required.

232. "From two given points on a circle to draw straight lines through a point $C$ in the circumference so they shall form with a line $MN$, given in position, a triangle $CMN$ of given area."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.



Let $A$ and $B$ be the two given points, $O$ the center of the circle. Draw $OH$, $AS$, $BR$ and $CK$ perpendicular to $MN$.

Let $OH = a$, $OA = b$, $OB = c$, $OC = r$, $\angle AOH = \beta$, $\angle BOH = \gamma$, $\angle COH = \theta$. Then is $AS = a - b\cos\beta$, $BR = a - c\cos\gamma$, $CK = a - r \times \cos\theta$, $SH = b\sin\beta$, $RH = c\sin\gamma$, $KH = r\sin\theta$.

By similar triangles we have

$$AS - CK : KH + SH :: CK : MK = \frac{(a - r\cos\theta)(r\sin\theta + b\sin\beta)}{r\cos\theta - b\cos\beta},$$

$$BR - CK : KH - RH :: CK : NK = \frac{(a - r\cos\theta)(r\sin\theta - c\sin\gamma)}{r\cos\theta - c\cos\gamma}.$$

$$\therefore MN = \frac{(a - r\cos\theta)(r\sin\theta + b\sin\beta)}{r\cos\theta - b\cos\beta} - \frac{(a - r\cos\theta)(r\sin\theta - c\sin\gamma)}{r\cos\theta - c\cos\gamma},$$

and area $CMN = \frac{1}{2} MN \times CK$

$$= \frac{(a - r\cos\theta)^2(r\sin\theta + b\sin\beta)}{2(r\cos\theta - b\cos\beta)} - \frac{(a - r\cos\theta)^2(r\sin\theta - c\sin\gamma)}{2(r\cos\theta - c\cos\gamma)}$$

$$= m. \quad (1)$$

Reducing (1), we get an equation of the sixth degree in $\cos\theta$.

[Professors Scheffer and Kershner, and Mr. Eastwood, each locate the points $A$ and $B$ on the *circumference* of the circle, and thereby effect the solution by an equation of the fourth degree; but this is a restriction of the language in which the question is announced, and cannot be regarded as a general solution; for the circumference of a circle is not the circle, but is a *line* which envelopes the circle.—Ed.]

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233. "If $ABCD$ be a spherical quadrilateral whose sides $AB$, $DC$ are produced to meet at $P$, and $AD$, $BC$, to meet at $Q$, and whose diagonals $AC$, $BD$ intersect at $R$, then

$$\sin AB \sin CD \cos P - \sin AD \sin BC \cos Q = \pm \sin AC \sin BD \cos R."$$

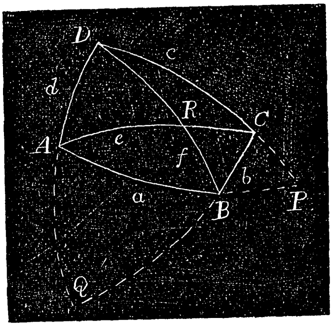
### SOLUTION BY PROF. J. SCHEFFER, MERCERSBERG, PA.

Denoting, for the sake of brevity, $AB$, $BC$, $CD$, $DA$, $AC$, $BD$, respectively, by $a$, $b$, $c$, $d$, $e$, $f$; and by $A$, $B$, $C$, $D$, the four angles of the quadrilateral, we have, in the $\triangle ABC$,

$$\sin a : \sin e = \sin ACB : \sin B,$$

and in $\triangle BCD$, $\quad \sin c : \sin f = \sin DBC : \sin C$; $\therefore$ by multiplying,

$$\sin a \sin c \sin B \sin C = \sin e \sin f \sin ACB \sin DBC. \quad (1)$$



Again, in the triangle $ABC$, we have

$$\sin b : \sin e = \sin CAB : \sin B,$$

and in the triangle $ABD$ we have

$$\sin d : \sin f = \sin ABD : \sin A. \quad \therefore$$

$$\sin b \sin d \sin A \sin B = \sin e \sin f \sin CAB \times \sin ABD. \quad (2)$$

But, in the triangle $BCP$,

$$\sin B \sin C = \frac{\cos P + \cos B \cos C}{\cos b}, \quad (3)$$

and, in $\triangle ABQ$,

$$\sin A \sin B = \frac{\cos Q + \cos A \cos B}{\cos a}, \quad (4)$$

in $\triangle BCR$,

$$\sin ACB \sin DBC = \frac{\cos R + \cos ACB \cos DBC}{\cos b}, \quad (5)$$

where $\angle BRC = R$; and, in the triangle $ABR$ we have

$$\sin CAB \sin ABD = \frac{-\cos R + \cos CAB \cos ABD}{\cos a}. \quad (6)$$

Substituting these values in (1) and two, we obtain

$$\sin a \sin c\, (\cos P + \cos B \cos C) = \sin e \sin f (\cos R + \cos ACB \cos DBC), \quad (7)$$

$$\sin b \sin d\, (\cos Q + \cos A \cos B) = \sin e \sin f (-\cos R + \cos CAB \cos ABD). \quad (8)$$

Substituting in (8) from (7), we get

$$\begin{aligned} \sin a \sin c \cos P - \sin b \sin d \cos Q &= 2 \sin e \sin f \cos R \\ &\quad + \sin e \sin f (\cos ACB \cos DBC - \cos CAB \cos DBA) \\ &\quad - \sin a \sin c \cos B \cos C + \sin b \sin d \cos A \cos B. \end{aligned} \quad (9)$$

Substituting in (9)

$$\cos ACB = \frac{\cos a - \cos b \cos e}{\sin b \sin e}, \quad \cos DBC = \frac{\cos c - \cos b \cos f}{\sin b \sin f},$$

$$\cos CAB = \frac{\cos b - \cos a \cos e}{\sin a \sin e}, \quad \cos DBA = \frac{\cos d - \cos a \cos f}{\sin a \sin f};$$

$$\cos A = \frac{\cos f - \cos a \cos d}{\sin a \sin d}, \quad \cos B = \frac{\cos e - \cos a \cos b}{\sin a \sin b},$$

$$\cos C = \frac{\cos f - \cos b \cos c}{\sin b \sin c};$$

and we get, after some transformations that present no difficulty,

$$\sin a \sin c \cos P - \sin b \sin d \cos Q = 2 \sin e \sin f \cos R + \cos a \cos c - \cos b \cos d. \quad (10)$$

But

$$\cos a = \cos AR \cos BR - \sin AR \sin BR \cos R,$$

$$\cos c = \cos DR \cos CR - \sin DR \sin CR \cos R,$$

$$\cos b = \cos BR \cos CR + \sin BR \sin CR \cos R,$$
$$\cos d = \cos AR \cos DR + \sin AR \sin DR \cos R.$$

Consequently

$$\begin{aligned}\cos a \cos c - \cos b \cos d &= -(\sin AR \cos CR + \sin CR \cos AR) \\ &\qquad \times(\sin BR \cos DR + \sin DR \cos BR) \cos R \\ &= -\sin (AR + CR) \sin (BR + DR) \cos R \\ &= -\sin e \sin f \cos R.\end{aligned}$$

Therefore, $\sin a \sin c \cos P - \sin b \sin d \cos Q = \pm \sin e \sin f \cos R$, the double sign referring to the case of $R$ being acute or obtuse.

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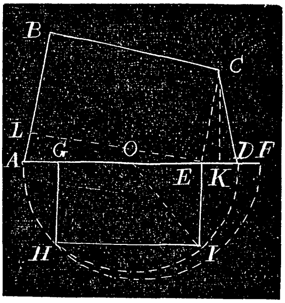
234. "$ABCD$ is a trapezium, $AB = a$, $CD = b$, $AD = c$; angle $ABC$ is a right angle, and $E$ is a point on $AD$ such that angle $BCE$ is a right angle and $CE = CD = b$."

It is required to find $BC$, $DE$ and $AE$.

SOLUTION BY PROF. M. C. STEVENS, LA FAYETTE, INDIANA.



*Construction.*—On $AF$ take $AG = a - b$, $GF = 2b$ and $AD = c$. On $AF$ and $AD$ describe semicircumferences. Draw $GH$ perpendicular to $AF$ meeting the circumference on $AF$ in $H$. Draw $HI$ parallel to $AF$ meeting the semicircumference $AD$ in $I$. Let fall $IE$ perpendicular to $AD$. On $AE$ construct the triangle $AEL$ right angled at $L$, with $AL = AG$. Complete the parallelogram $LBCE$, and join $CD$; then is $ABCD$ the trapezium required.

*Demonstration.*—Bisect $ED$ in $K$ and join $CK$. By construction, $AG \times GF = GH^2 = EI^2 = AE \times ED$; $\therefore GF : ED :: AE : AG$; or $\frac{1}{2}GF : \frac{1}{2}ED :: AE : AG$. Since $AB = a$ and $AL = AG = a - b$, $\therefore LB = EC = b$. Also $\frac{1}{2}GF = b$. Whence the above proportion may be written $CE : EK :: AE : AL$. But the angle $CEK = LAE$, $\therefore$ the triangle $CEK$ is similar to $EAL$. Hence, the angle $CKE$ is a right angle and $CD = CE = b$.

*Calculation.*—$OE = \sqrt{[\frac{1}{4}c^2 - 2b(a-b)]}$, $\therefore AE = \frac{1}{2}c \pm \sqrt{[\frac{1}{4}c^2 - 2b(a-b)]}$ and $DE = \frac{1}{2}c \mp \sqrt{[\frac{1}{4}c^2 - 2b(a-b)]}$; $BC = EL$

$$= \sqrt{\left\{ (\tfrac{1}{2}c \pm \sqrt{[\tfrac{1}{4}c^2 - 2b(a-b)]})^2 - (a-b)^2 \right\}}.$$

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235. "Find the condition that the general equation of the $n$th degree may have $q$ equal roots."

### SOLUTION BY PROF. J. H. KERSHNER, MERCERSBERG, PA.

The absolute term is the product of all the roots with their signs changed, the last coefficient is the sum of the products of every $n$—1 roots with their signs changed, etc. Therefore, if $a$ be one of the $q$ equal roots, the absolute term must be divisible by $a^q$, the last coefficient must be divisible by $a^{q-1}$, etc. The form of the general equation is as follows:

$$x^n+p_1x^{n-1}+\ldots m_1ax^{q-1}+m_2a^2x^{q-2}+\ldots m_{q-1}\,a^{q-1}\,x+m_qa^q=0.$$

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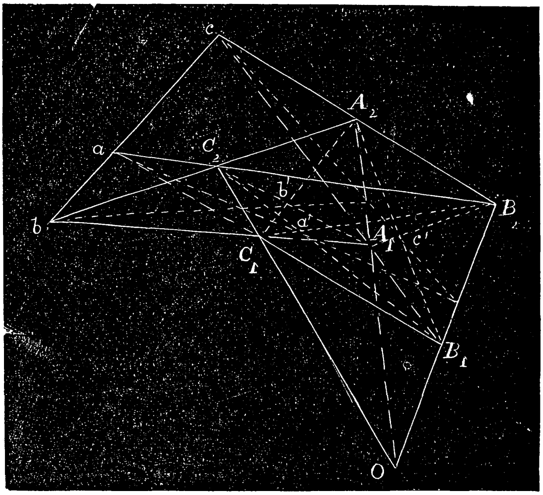
236. "If $A_1A_2$, $B_1B_2$, $C_1C_2$ are three lines which meet in a point $O$, then there are four different ways in which the points $A_1$, $B_1$, $C_1$, $A_2$, $B_2$, $C_2$, can be combined into two homologous triangles, and for each combination there is a different axis of homology. Show that these four axes form a complete quadrilateral whose diagonals intersect each other on the lines $A_1A_2$, $B_1B_2$, $C_1C_2$."

### SOLUTION BY PROF. JOHNSON.

The fundamental property of homologous triangles is demonstrated in my solution of problem 186, ANALYST, Vol. V, No. 1, by a method which is readily applied to the present problem. Thus, let the line $OA_1$, $OB_1$, $OC_1$ be at first supposed not in the same plane; then $a$ being the intersection of $B_1C_1$ with $B_2C_2$ is at once in the plane $A_1B_1C_1$ and in the plane $A_2B_2C_1$, hence it is in the intersection of these planes: for a similar reason $b$ and $c$ are in the intersection of the same planes, therefore $a\,b\,c$ is a straight line, and in the projection of the figure upon any plane $a\,b\,c$ is likewise a straight line. It may be remarked that the figure in space is simply that formed by the intersections of five planes, and hence may be called a *complete pentahedron.* The full and broken lines in the figure represent a complete pentahedron; it has ten edges and ten vertices each of which is the intersection of three planes which cut



out two triangles on the other two planes, and in any projection of the figure, the triangles become homologous triangles, the point in which the three planes meet becomes the centre, and the intersection of the planes of the triangles the axis of homology.

Now let $c'$ be the intersection of $A_1B_2$ with $A_2B_1$ and let $a'$ and $b'$ be determined in like manner; then will $a'b'c$, $a'bc'$ and $ab'c'$ be straight lines, the six points $a\,b\,c\,a'b'c'$ are in a plane and form the vertices of a complete quadrilateral of which $aa'$, $bb'$ and $cc'$ are the diagonals. But $cc'$ is in the plane $OA_1B_1$ and therefore intersects $OA_1$, and $bb'$ is in the plane $OA_1C_1$ and therefore intersects $OA_1$, hence $cc'$ and $bb'$ meet in the point in which the plane of the complete quadrilateral cuts the line $A_1A_2$. Hence also in the plane figure which is the projection of the figure in space, the diagonals $bb'$ and $cc'$ meet on the line $A_1A_2$.

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237. "Prove that

$$\int_0^\pi \frac{\cos\frac{1}{2}(n-1)(\theta+\pi)\sin\frac{1}{2}[n(\theta+\pi)]}{\sin\frac{1}{2}(\theta+\pi)}d\theta=\pi."$$

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.

In Todhunter's Plane Trigonometry, page 245, 4th Ed., we have

$$\frac{\cos\left[a+\frac{1}{2}(n-1)(\beta+\pi)\right]\sin\frac{1}{2}n(\beta+\pi)}{\sin\frac{1}{2}(\beta+\pi)}=\cos a-\cos(a+\beta)+\cos(a+2\beta)$$
$$-\ldots+(-)^{n-1}\cos[a+(n-1)\beta].$$

Let $a=0$, $\beta=\theta$; then

$$\frac{\cos\left[\frac{1}{2}(n-1)(\theta+\pi)\right]\sin\frac{1}{2}(\theta+\pi)}{\sin\frac{1}{2}(\theta+\pi)}=\cos 0-\cos\theta+\cos 2\theta-\&c.$$

$$\therefore\int_0^\pi\frac{\cos\left[\frac{1}{2}(n-1)(\theta+\pi)\right]\sin\frac{1}{2}n(\theta+\pi)}{\sin\frac{1}{2}(\theta+\pi)}\,d\theta$$
$$=\int_0^\pi(\cos 0-\cos\theta+\cos 2\theta-\&c.)d\theta$$
$$=\Big[\theta-\sin\theta+\tfrac{1}{2}\sin 2\theta-\&c.\Big]_0^\pi$$
$$=\pi.$$

[The solutions of problems 238, 239 and 240, will be published in No. 2 as follows:—Solution of 238, by Artemas Martin, 239, a solution by Chas. H. Kummell and also a solution by R. J. Adcock, 240, solution by E. B. Seitz..]